**Quadratics (1)**

**1.**  **(a)** State the relation between $a,b and c$ such that the equation $ax^{2}+bx+c=0$ ($a\ne 0)$ has equal roots.

 **(b)** If the equation $a^{2}x^{2}+3abx+ac+2b^{2}=0$ ($a\ne 0)$ has equal roots,

 show that the roots for the equation $ac\left(x+1\right)^{2}=b^{2}x$ ($a,b,c\ne 0)$ are equal.

**(a)** The condition for the equation $ax^{2}+bx+c=0$ has equal roots is $∆=b^{2}-4ac=0$

**(b)** If the equation $a^{2}x^{2}+3abx+2b^{2}=0$ has equal roots,

$$∆=\left(3ab\right)^{2}-4a^{2}\left(ac+2b^{2}\right)=0 \left(a\ne 0\right)$$

 $∴b^{2}=4ac$

 **Method 1**

 For the equation $ac\left(x+1\right)^{2}=b^{2}x⟺ac\left(x^{2}+2 x+1\right)-b^{2}x=0$

$$⟺acx^{2}+\left(2ac-b^{2}\right)x+ac=0$$

 Since $∆=\left(2ac-b^{2}\right)^{2}-4\left(ac\right)\left(ac\right)=b^{4}-4 a c b^{2}=b^{2}\left(b^{2}-4ac\right)=0$

 Therefore $ac\left(x+1\right)^{2}=b^{2}x$ has equal roots.

 **Method 2**

Since$a,b,c\ne 0$

$$ac\left(x+1\right)^{2}=b^{2}x ⟺ ac\left(x+1\right)^{2}=4acx ⟺ \left(x+1\right)^{2}= 4x⟺ x^{2}+2 x+1=4x⟺\left(x-1\right)^{2}=0$$

 $∴ x=1$ (equal roots)

**2.** If x is real, show that the expression $y=\frac{x^{2}+x+1}{x+1}$ does not have a value between $-3 and 1$ .

 $y=\frac{x^{2}+x+1}{x+1} ⟺ y\left(x+1\right)=x^{2}+x+1 ⟺x^{2}+\left(1-y\right)x+\left(1-y\right)=0$

 Since x is real, $∆=\left(1-y\right)^{2}-4\left(1-y\right)\geq 0$

$$y^{2}+2 y-3\geq 0$$

$$\left(y-1\right) \left(y+3\right)\geq 0$$

 $\left\{\begin{array}{c}y-1\leq 0\\y+3\leq 0\end{array} or \right.\left\{\begin{array}{c}y-1\geq 0\\y+3\geq 0\end{array} \right.$

$$∴y\leq -3 or y\geq 1$$

**3.**  Let the equations $x^{2}+ax+b=0$ and $x^{2}+cx+d=0$ ($b\ne d)$ have one non-zero common root.

 Form an equation with the other roots of these equations.

 Let $α$ be the non-zero common root. Then by Vieta Theorem the other roots are $\frac{b}{α} and \frac{d}{α}$ .

 Also $α^{2}+aα+b=0$ and $α^{2}+cα+d=0$ and their difference is

 $\left(a-c\right)α+\left(b-d\right)=0$ giving $α=\frac{d-b}{a-c}$ (check that $a\ne c$)

 Thus, the other roots are $\frac{b\left(a-c\right)}{d-b}$ and $\frac{d\left(a-c\right)}{d-b}$.

 Sum of roots = $\frac{b\left(a-c\right)}{d-b}+\frac{d\left(a-c\right)}{d-b}=\frac{\left(b+d\right)\left(a-c\right)}{d-b}$

 Product of roots = $\frac{b\left(a-c\right)}{d-b}×\frac{d\left(a-c\right)}{d-b}=\frac{bd\left(a-c\right)}{\left(d-b\right)^{2}}$

 Therefore the required equation is $x^{2}-\frac{\left(b+d\right)\left(a-c\right)}{d-b}x+\frac{bd\left(a-c\right)}{\left(d-b\right)^{2}}=0$

**4.** If $a,b and c$ are real numbers, show that the roots of the equation $\left(a-b-c\right)x^{2}+ax+b+c=0$ is real.

 If one of the roots is twice the other, show that $b+c=\frac{a}{3} or \frac{2a}{3}$ .

 For the equation $\left(a-b-c\right)x^{2}+ax+b+c=0$

 $∆=a^{2}-4\left(a-b-c\right)\left(b+c\right)=a^{2}-4\left[a-\left(b+c\right)\right]\left(b+c\right)=a^{2}-4a\left(b+c\right)+4\left(b+c\right)^{2}$

 $=\left[a-2\left(b+c\right)\right]^{2}\geq 0$

 Hence, the roots of the equation $\left(a-b-c\right)x^{2}+ax+b+c=0$ is real.

 If one of the roots is twice the other, let $α, 2α$ be the roots.

 By Vieta Theorem, $α+2α=-\frac{a}{a-b-c} …(1)$

 $\left(α\right)\left(2α\right)=\frac{b+c}{a-b-c}$ …(2)

 From (1), $α=-\frac{a}{3\left[a-\left(b+c\right)\right]} … (3)$

 From (2), $2α^{2}=\frac{b+c}{a-\left(b+c\right)} …(4)$

 $\left(3\right)\downright \left(4\right), 2\left\{-\frac{a}{3\left[a-\left(b+c\right)\right]}\right\}^{2}=\frac{b+c}{a-\left(b+c\right)}$

 $\frac{2a^{2}}{9\left[a-\left(b+c\right)\right]}=b+c$

 $2a^{2}=9a\left(b+c\right)-9(b+c)^{2}$

 $9(b+c)^{2}-9a\left(b+c\right)+2a^{2}=0$

 $\left[3\left(b+c\right)-a\right]\left[3\left(b+c\right)-2a\right]=0$

 $∴ b+c=\frac{a}{3} or \frac{2a}{3}$

**5.** If $α\_{1} and β\_{1}$ are the roots of the equation $x^{2}+2ax+b^{2}=0$ and $α\_{2} and β\_{2}$ are the roots of the equation $x^{2}+2cx+d^{2}=0$ , show that :

 **(a)** If $α\_{1}+α\_{2}= β\_{1}+β\_{2}$ , then $a^{2}+d^{2}=b^{2}+c^{2}$ ,

 **(b)** If $α\_{1}α\_{2}+ β\_{1}β\_{2}=0$ , then $b^{2}d^{2}=a^{2}d^{2}+b^{2}c^{2}$.

 **(a)** By Vieta Theorem, $α\_{1}+β\_{1}=-2a, α\_{1}β\_{1}=b^{2}$ and $α\_{2}+β\_{2}=-2c, α\_{2}β\_{2}=d^{2}$

 Since $α\_{1}+α\_{2}= β\_{1}+β\_{2}$

 $α\_{1}- β\_{1}=-\left(α\_{2}-β\_{2}\right)$

 $\left(α\_{1}- β\_{1}\right)^{2}=\left(α\_{2}-β\_{2}\right)^{2}$

 $α\_{1}^{2}-2 α\_{1} β\_{1}+β\_{1}^{2}=α\_{2}^{2}-2 α\_{2} β\_{2}+β\_{2}^{2}$

 $\left(α\_{1}+ β\_{1}\right)^{2}-4 α\_{1} β\_{1}=\left(α\_{2}+β\_{2}\right)^{2}-4α\_{2} β\_{2}$

 $\left(-2a\right)^{2}-4 b^{2}=\left(-2c\right)^{2}-4d^{2}$

 $a^{2}+d^{2}=b^{2}+c^{2}$

 **(b)** $a^{2}d^{2}+b^{2}c^{2}=\left(-\frac{α\_{1}+β\_{1}}{2}\right)^{2}\left(α\_{2}β\_{2}\right)+\left(α\_{1}β\_{1}\right)\left(-\frac{α\_{2}+β\_{2}}{2}\right)^{2}=\frac{α\_{1}^{2}+2 α\_{1} β\_{1}+β\_{1}^{2}}{4}\left(α\_{2}β\_{2}\right)+\left(α\_{1}β\_{1}\right)\frac{α\_{2}^{2}+2 α\_{2} β\_{2}+β\_{2}^{2}}{4}$

 $=\frac{1}{4}\left(4 α\_{1}α\_{2} β\_{1}β\_{2}+α\_{1}β\_{1} α\_{2}^{2}+α\_{1} β\_{1} β\_{2}^{2}+α\_{2}β\_{2} α\_{1}^{2}+α\_{2} β\_{2} β\_{1}^{2}\right)$

 $=\frac{1}{4}\left[4 α\_{1}α\_{2} β\_{1}β\_{2}+\left(α\_{1}α\_{2}+ β\_{1}β\_{2}\right)α\_{1}β\_{2}+\left(α\_{1}α\_{2}+ β\_{1}β\_{2}\right)α\_{2}β\_{1}\right]=\frac{1}{4}\left[4 α\_{1}α\_{2} β\_{1}β\_{2}+\left(0\right)α\_{1}β\_{2}+\left(0\right)α\_{2}β\_{1}\right]$

 $=α\_{1}α\_{2} β\_{1}β\_{2}=\left(α\_{1}β\_{1}\right)\left(α\_{2}β\_{2}\right)=b^{2}d^{2}$

**6.** If the equation $ax^{2}+bx+c=0$ ($a\ne 0)$ has real roots, show that the equation

$$\left(a+c-b\right)x^{2}-2\left(a-c\right)x+\left(a+c+b\right)=0$$

 has also real roots.

 Show that if $α$ and $β$ are the roots for the first equation, then the product of roots of the second equation is

$$\frac{\left(1-α\right)\left(1-β\right)}{\left(1+α\right)\left(1+β\right)}$$

 The equation $ax^{2}+bx+c=0$ ($a\ne 0)$ has real roots if and only if $b^{2}-ac\geq 0$

 For the equation $\left(a+c-b\right)x^{2}-2\left(a-c\right)x+\left(a+c+b\right)=0$ ,

 $∆=\left[-2\left(a-c\right)\right]^{2}-4\left(a+c-b\right)\left(a+c+b\right)$

 $=4\left(a-c\right)^{2}-4\left[\left(a+c\right)^{2}-b^{2}\right]$

 $=4\left[\left(a-c\right)^{2}-\left(a+c\right)^{2}\right]+4b^{2}$

 $=4\left(-4ac\right)+4b^{2}$

 $=4\left(b^{2}-4ac\right)\geq 0$

 Therefore the second equation has real roots.

 For the first equation, by Vieta Theorem, $α+ β=-\frac{b}{a} $ , $α β=\frac{c}{a}$

 The product of roots of the second equation is

 $\frac{a+c+b}{a+c-b}=\frac{1+\frac{c}{a}+\frac{b}{a}}{1+\frac{c}{a}-\frac{b}{a}}=\frac{1+α β-\left(α+ β\right)}{1+α β+\left(α+ β\right)}=\frac{\left(1-α\right)\left(1-β\right)}{\left(1+α\right)\left(1+β\right)}$

**7.** If $α,β$ are roots of the equation $x^{2}+px+q=0$ and $α\_{1},β\_{1}$ are roots of the equation $x^{2}+p\_{1}x+q\_{1}=0$.

 Express $\left(α-α\_{1}\right)\left(α-β\_{1}\right)+\left(β-α\_{1}\right)\left(β-β\_{1}\right)$ in terms of $p,q,p\_{1} and q\_{1}$.

 By Vieta Theorem, $α+ β=-p $ , $α β=q$

$α\_{1}+β\_{1}=-p\_{1}, α\_{1}β\_{1}=q\_{1}$

 $\left(α-α\_{1}\right)\left(α-β\_{1}\right)+\left(β-α\_{1}\right)\left(β-β\_{1}\right)$

 $=α^{2}+β^{2}-α \left(α\_{1}+ β\_{1}\right)-β \left(α\_{1}+ β\_{1}\right)+2 α\_{1} β\_{1}$

 $=\left(α+ β\right)^{2}-2α β-\left(α+ β\right)\left(α\_{1}+ β\_{1}\right)+2 α\_{1} β\_{1}$

 =$\left(-p\right)^{2}-2q-\left(-p\right)\left(-p\_{1}\right)+2 q\_{1}$

 $=p^{2}-p p\_{1}-2 q+2 q\_{1}$

**8.** Show that the expression $\frac{5}{2x^{2}+3x+3}$ is positive and find its greatest value.

 Hence find the smallest values of $\frac{6x^{2}+9x+4}{2x^{2}+3x+3}$ .

 Sketch the functions of $2x^{2}+3x+3, \frac{5}{2x^{2}+3x+3}, \frac{6x^{2}+9x+4}{2x^{2}+3x+3}$ together on the same graph.

 $2x^{2}+3x+3=2\left[x^{2}+\frac{3}{2}x+\frac{3}{2}\right]=2\left[x^{2}+2\left(\frac{3}{4}\right)x+\left(\frac{3}{4}\right)^{2}-\left(\frac{3}{4}\right)^{2}+\frac{3}{2}\right]=2\left[\left(x+\frac{3}{2}\right)^{2}-\frac{9}{16}+\frac{3}{2}\right]$

 $=2\left[\left(x+\frac{3}{4}\right)^{2}+\frac{15}{16}\right]>0$

 Therefore $\frac{5}{2x^{2}+3x+3}$ is positive.

 $Min\left\{2x^{2}+3x+3\right\}=Min\left\{2\left[\left(x+\frac{3}{4}\right)^{2}+\frac{15}{16}\right]\right\}=2\left[0+\frac{15}{16}\right]=\frac{15}{8}$

 $∴Max\left\{\frac{5}{2x^{2}+3x+3}\right\}=\frac{5}{\frac{15}{8}}=\frac{8}{3}$

 $\frac{6x^{2}+9x+4}{2x^{2}+3x+3}=\frac{3\left(2x^{2}+3x+3\right)-5}{2x^{2}+3x+3}=3-\frac{5}{2x^{2}+3x+3}$

 $∴Min\left\{\frac{6x^{2}+9x+4}{2x^{2}+3x+3}\right\}=3-\frac{8}{3}=\frac{1}{3}$



**Yue Kwok Choy**

**4/9/2016**